A note on a result of Liptser-Shiryaev

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Abstract

Given two stochastic equations with different drift terms, under very weak assumptions Liptser and Shiryaev provide the equivalence of the laws of the solutions to these equations by means of Girsanov transform. Their assumptions involve both the drift terms. We are interested in the same result but with the main assumption involving only the difference of the drift terms. Applications of our result will be presented in the finite as well as in the infinite dimensional setting.

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1 Introduction

Let us consider the Itô equation

$$dX(t) = b(t, X(t))\ dt + \sigma(t, X(t))\ dW(t), \quad X(0) = x. \tag{1.1}$$

If we know that there exists a solution, we ask about uniqueness and characterization of its law. We can look at equation (1.1) as a modification of equation

$$dZ(t) = a(t, Z(t))\ dt + \sigma(t, Z(t))\ dW(t), \quad Z(0) = x. \tag{1.2}$$

by a change of the drift term. Equation (1.2) is a "good" reference equation, for which existence and uniqueness hold true. Since these two equations differ only in the drift terms, a classical tool to study equation (1.1) is the Girsanov transform.

In [11], Chapter 7 is devoted to this problem, where Liptser and Shiryaev investigate the relation between the laws of processes solving equations (1.1) and (1.2). In this paper, we address the same problem.

As far as the results are concerned, first in dimension one we prove results similar to [11] but with different assumptions; in fact, our hypotheses involve the difference $b - a$ whereas in [11] involve separately a and b. Then, we consider the case of dimension bigger than one. Our analysis includes the uniqueness

problem, not tackled in [11]. Moreover we extend these results to the infinite dimensional setting, whereas [11] deals only with the finite dimensional case. Here, when we say finite dimensional we mean that the state space is finite dimensional, i.e. the unknown X is a vector process with a finite number ($d < \infty$) of components; this models stochastic differential equations on the state space $\mathbb{R}^d$. However, the infinite dimensional setting is related to abstract models of stochastic partial differential equations (see, e.g., the book by Da Prato and Zabczyk [3]). Actually, the infinite dimensional setting is one of the main motivations of our study, as it will be explained in Section 9.

As far as the techniques are concerned, in some parts our proofs are shorter than in [11], in the sense that even with the same assumption of [11] we get the results of [11] with shorter proofs.

Now, we explain how the paper is organized. We start our exposition with the one dimensional setting. Extension to dimension bigger than one is in Section 8. After the basic results presented in Sections 2 and 3, we shall analyze uniqueness in law in Section 4, the absolute continuity in Section 5 and the equivalence of the laws in Section 6. In Section 7 our results will be compared with those in [11]. In the final section the novelty of our results will be discussed, also in the infinite dimensional setting.

2 Preliminaries

We set our problem as in the book of Liptser and Shiryaev [11], that is in a setting more general than (1.1)-(1.2).

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space and $\{\mathbb{F}_t\}_{t\geq 0}$ a filtration. We will always assume that the probability space is complete and the filtration is right continuous. We denote by $\mathbb{E}$ the expectation with respect to the measure $\mathbb{P}$, and by $\mathbb{F}_T(X)$ the σ-algebra generated by $\{X(u), 0 \leq u \leq T\}$.

When dealing with a Polish space, i.e. a complete separable metric space, the σ-algebra associated is the Borel σ-algebra. In particular, for $0 < t \leq T$ let $\mathcal{B}_t$ be the σ-algebra of Borelian subsets of $C([0,t];\mathbb{R})$. We say that a measurable functional $\phi : [0,T] \times C([0,T];\mathbb{R}) \to \mathbb{R}$ is non anticipative if, for each $t \in [0,T]$, $\phi(t,\cdot)$ is $\mathcal{B}_t$-measurable.

The two equations to deal with are

$$dX(t) = b(t,X)\ dt + \sigma(t,X)\ dW(t); \qquad X(0) = x \tag{2.1}$$

$$dZ(t) = a(t,Z)\ dt + \sigma(t,Z)\ dW(t); \qquad Z(0) = x \tag{2.2}$$

Here, a, b and σ are non anticipative measurable functionals. W is a Wiener process with respect to the stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P})$.

We need to recall what is a weak or strong solution. We consider processes X with a.e. path in $C([0,T];\mathbb{R})$, which are adapted to the filtration $\{\mathbb{F}_t\}_{t\geq 0}$ and solve equation (2.1) a.s.:

$$X(t) = x + \int_0^t b(s,X)\ ds + \int_0^t \sigma(s,X)\ dW(s) \qquad \mathbb{P} - a.s. \tag{2.3}$$

for every $t \in [0,T]$. It is necessary that

$$\mathbb{P}\{\int_0^T |b(s,X)|ds < \infty\} = \mathbb{P}\{\int_0^T \sigma(s,X)^2 ds < \infty\} = 1.$$

For simplicity, we fix the initial data $x \in \mathbb{R}$; however, our results can be extended to cover the case of random initial data.

Definition 2.1 (weak solution) *We say that there exists a weak solution to equation* (2.1) *if there exist a stochastic basis* $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$*, an* $\{\mathbb{F}_t\}$*-Wiener process* W *and an* $\{\mathbb{F}_t\}$*-adapted process* X *defined in it such that* X *solves equation* (2.1) $\mathbb{P}$*-a.s.*
We denote this solution by the triplet $\big(X,\ (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$.

On the other hand, if X solves (2.1) on a (a priori) given stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ with a given Wiener process W, we have a strong solution. Therefore the Wiener process and the filtration are not part of the solution itself but are assigned.

Definition 2.2 (strong solution) *We say that there exists a strong solution to equation* (2.1) *if, given any stochastic basis* $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ *and* $\{\mathbb{F}_t\}$*-Wiener process* W*, there exists an* $\{\mathbb{F}_t\}$*-adapted process* X *such that* X *solves equation* (2.1) $\mathbb{P}$*-a.s.*

Moreover, we have two kinds of uniqueness.

Definition 2.3 (uniqueness in law) *We say that uniqueness in law holds for equation* (2.1) *if any two processes solving equation* (2.1) *with the same initial data have the same law.*

Definition 2.4 (pathwise uniqueness) *We say that pathwise uniqueness holds for equation* (2.1) *if given two processes* X *and* X' *solving equation* (2.1) *with the same initial data and defined with respect to the same stochastic basis* $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ *and Wiener process, we have* $\mathbb{P}\{X(t) = X'(t) \text{ for all } t\} = 1$.

In the following we shall assume that equation (2.2) has a unique strong solution; uniqueness has to be understood as pathwise uniqueness. But, a result of Cherny (see [2]) says that uniqueness in law, together with the strong existence, guarantees the pathwise uniqueness. Hence, we could simply assume existence of a strong solution and uniqueness in law.
On the other hand, from now on saying uniqueness of a weak solution we will mean uniqueness in law, unless otherwise specified.

Therefore, the coefficients a and σ are required to satisfy the usual growth

and Lipschitz conditions (see, e.g., [11]), that is

$$[\mathbf{A1}] \left[\begin{array}{l} \exists \text{ constants } L_1, L_2 \text{ and a function } K \text{ non decreasing and right continuous,} \\ \text{with } 0 \le K(s) \le 1, \text{ such that} \\ a(t,Y)^2 + \sigma(t,Y)^2 \le L_1 \int_0^t [1 + Y(s)^2] dK(s) + L_2[1 + Y(t)^2] \\ \hfill \forall t \in [0,T], Y \in C([0,T];\mathbb{R}) \\ \text{and} \\ |a(t,Y_1) - a(t,Y_2)|^2 + |\sigma(t,Y_1) - \sigma(t,Y_2)|^2 \\ \qquad \le L_1 \int_0^t |Y_1(s) - Y_2(s)|^2 dK(s) + L_2 |Y_1(t) - Y_2(t)|^2 \\ \hfill \forall t \in [0,T], Y_1, Y_2 \in C([0,T];\mathbb{R}) \end{array} \right.$$

Moreover, the coefficients a, b and σ are such that

$$[\mathbf{A2}] \left[\begin{array}{l} \exists \text{ a measurable functional } \gamma \text{ which is non anticipative finite and such that} \\ \sigma(s,Y)\gamma(s,Y) = b(s,Y) - a(s,Y) \qquad \forall s \in [0,T], Y \in C([0,T];\mathbb{R}). \end{array} \right.$$

Few technical details: from now on, we consider only finite time intervals $[0,T]$. Then, the law of a process solving equation (2.1) or (2.2) is a probability measure on $\mathcal{B}_T$. Moreover, if there is uniqueness in law for an equation with drift term a we denote by μ^a this unique law (unless otherwise stated). If a measure ν_1 is absolutely continuous with respect to a measure ν_2 we write $\nu_1 \prec \nu_2$; if they are equivalent, i.e. $\nu_1 \prec \nu_2$ and $\nu_2 \prec \nu_1$, we write $\nu_1 \sim \nu_2$.

3 An easy case

In this section, we prove a result of equivalence of laws for equations (2.2) and (2.1) but in the particular case of $b = a + g$ with a strong assumption on σ and g. The proof is based on classical tools of Girsanov transform and Novikov condition.

Instead of equation (2.1), let us consider

$$dY(t) = a(t,Y)\, dt + g(t,Y)\, dt\ + \sigma(t,Y)\, dW(t), \qquad Y(0) = x, \tag{3.1}$$

where g is a non anticipative measurable functional. Moreover, we assume that there exists a finite and non anticipative measurable functional α such that

$$\sigma(s,Y)\alpha(s,Y) = g(s,Y) \tag{3.2}$$

for each $s \in [0,T]$ and $Y \in C([0,T];\mathbb{R})$.

Remark 3.1 *Relationship* (3.2) *is a compatibility condition; it means that when σ vanishes, also g must vanish. In this case, α may be chosen arbitrarily in order to satisfy* (3.2)*. But, as we shall see, in the Girsanov transform α takes into account the change of drift between equations* (3.1) *and* (2.2)*. Therefore we are interested only in the solution α of* (3.2) *which vanishes when $g = 0$, i.e. when the two drift terms are the same. Hence, from now on we consider*

$$\alpha(s,Y) = \sigma^+(s,Y)g(s,Y) \tag{3.3}$$

where

$$\sigma^+(s,Y) = \begin{cases} \dfrac{1}{\sigma(s,Y)}, & \sigma(s,Y) \neq 0 \\ 0, & \sigma(s,Y) = 0 \end{cases}$$

We have the following result.

Theorem 3.2 *Assume there exists a unique weak solution* $\big(Z,\ (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ *to equation* (2.2)*. If*

$$\sup_{X \in C([0,T];\mathbb{R})} \int_0^T \alpha(s,X)^2 ds = c < \infty, \tag{3.4}$$

then equation (3.1) *has a weak solution, which is unique in law. Moreover, the law of the process* Z *is equivalent to the law of the process solving* (3.1)*, that is* $\mu^a \sim \mu^{a+g}$*. In particular*

$$\frac{d\mu^{a+g}}{d\mu^a}(Z) = \mathbb{E}\Big[e^{\int_0^T \alpha(s,Z)dW(s) - \frac{1}{2}\int_0^T \alpha(s,Z)^2 ds} \big| \mathbb{F}_T(Z)\Big] \tag{3.5}$$

$\mathbb{P}$*-a.s.*

Proof. Because of (3.4) we have that

$$\mathbb{E}\Big[e^{\frac{1}{2}\int_0^T \alpha(s,Z)^2 ds}\Big] \leq e^{\frac{c}{2}} < \infty.$$

This is Novikov condition, which allows to apply Girsanov transform. More precisely (see [8]), Novikov condition makes sure that the process δ defined by

$$\delta_t = e^{\int_0^t \alpha(s,Z)dW(s) - \frac{1}{2}\int_0^t \alpha(s,Z)^2 ds},\ 0 \leq t \leq T,$$

is a martingale. To highlight the dependence on Z and W we will often write δ_T as $\delta_T(Z,W)$. We define a new probability measure on $(\Omega, \mathcal{F}_T)$ by $d\mathbb{P}^* = \delta_T(Z,W)d\mathbb{P}$. Then Girsanov theorem (see [6]) tells us that

$$W^*(t) = W(t) - \int_0^t \alpha(s,Z)\ ds\ , \qquad t \in [0,T],$$

is a Wiener process with respect to $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}^*)$; substituting into equation (2.2) we get

$$Z(t) = x + \int_0^t a(s,Z)\ ds + \int_0^t g(s,Z)\ ds + \int_0^t \sigma(s,Z)dW^*(s).$$

This means that $\big(Z,\ (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}^*),\ W^*\big)$ is a weak solution of equation (3.1).

For any Borelian subset Λ of $C([0,T];\mathbb{R})$, set $\mathcal{L}_Y(\Lambda) = \mathbb{P}^*\{Z \in \Lambda\}$ and $\mu^a(\Lambda) = \mathbb{P}\{Z \in \Lambda\}$. Then $\mathcal{L}_Y \prec \mu^a$, since $\mathbb{P}^* \prec \mathbb{P}$ by construction. Moreover, consider the random variable $\mathbb{E}[\delta_T(Z,W)|\mathbb{F}_T(Z)]$; it is $\mathbb{F}_T(Z)$-measurable and

therefore there exists a $\mathcal{B}_T$-measurable non negative function $D : C([0,T];\mathbb{R}) \to \mathbb{R}$ such that $D(Z(\omega)) = \mathbb{E}[\delta_T(Z,W)|\mathbb{F}_T(Z)](\omega)$ for $\mathbb{P}$-a.e. ω. Now, we have

$$\mathcal{L}_Y(\Lambda) = \mathbb{P}^*\{Z \in \Lambda\} = \int\limits_{\{Z\in\Lambda\}} \delta_T(Z,W)\ d\mathbb{P} = \int\limits_{\{Z\in\Lambda\}} \mathbb{E}\big[\delta_T(Z,W)|\mathbb{F}_T(Z)\big]\ d\mathbb{P}$$

$$= \int\limits_{\{Z\in\Lambda\}} D(Z)\ d\mathbb{P} = \int_\Lambda D(z)\ d\mu^a(z) \quad (3.6)$$

Hence

$$\frac{d\mathcal{L}_Y}{d\mu^a}(Z) = D(Z) \qquad \text{for } Z \in C([0,T];\mathbb{R}).$$

This proves (3.5), as soon as we have uniqueness in law for equation (3.1).

Viceversa, any weak solution $\big(Y,(\tilde\Omega,\tilde{\mathbb{F}},\{\tilde{\mathbb{F}}_t\},\tilde{\mathbb{P}}),\tilde W\big)$ of equation (3.1) gives rise to a weak solution $\big(Y,(\tilde\Omega,\tilde{\mathbb{F}},\{\tilde{\mathbb{F}}_t\},\tilde{\mathbb{P}}^*),\tilde W^*\big)$ of equation (2.2), with a similar expression of the Radon-Nikodym derivative (only a change of sign appears). Indeed, thanks to (3.4)

$$\hat\delta_t(Y,\tilde W) = e^{-\int_0^t \alpha(s,Y)d\tilde W(s) - \frac{1}{2}\int_0^t \alpha(s,Y)^2 ds} \quad (3.7)$$

is a martingale; define $d\tilde{\mathbb{P}}^* = \hat\delta_T(Y,\tilde W)d\tilde{\mathbb{P}}$ and $\tilde W^*(t) = \tilde W(t) + \int_0^t \alpha(s,Y)\ ds$. Then, $\tilde W^*$ is a Wiener process with respect to $\tilde{\mathbb{P}}^*$ and

$$\mu^a(\Lambda) = \tilde{\mathbb{P}}^*\{Y\in\Lambda\} = \int\limits_{\{Y\in\Lambda\}} \hat\delta_T(Y,\tilde W)\ d\tilde{\mathbb{P}}. \quad (3.8)$$

Now, suppose there exist two different weak solutions of equation (3.1):

$$\big(Y_i,(\tilde\Omega_i,\tilde{\mathbb{F}}_i,\{\tilde{\mathbb{F}}_{i_t}\},\tilde{\mathbb{P}}_i),\tilde W_i\big) \qquad i=1,2.$$

We have that $d\tilde{\mathbb{P}}_i^* = \hat\delta_T(Y_i,\tilde W_i)d\tilde{\mathbb{P}}_i$; moreover,

$$\begin{aligned}\hat\delta_t(Y_i,\tilde W_i) &= e^{-\int_0^t \alpha(s,Y_i)d\tilde W_i(s) - \frac{1}{2}\int_0^t \alpha(s,Y_i)^2 ds}\\ &= e^{-\int_0^t \alpha(s,Y_i)d\tilde W_i^*(s) + \frac{1}{2}\int_0^t \alpha(s,Y_i)^2 ds} =: \frac{1}{\underline{\delta}_t(Y_i,\tilde W_i^*)}.\end{aligned}$$

Again (3.4) provides that $\underline{\delta}_T(Y_i,\tilde W_i^*)$ is well defined. Then, $d\tilde{\mathbb{P}}_i = \underline{\delta}_T(Y_i,\tilde W_i^*)d\tilde{\mathbb{P}}_i^*$. Now, uniqueness in law for the solution of equation (2.2) means that the joint distribution of (Y_1,W_1^*) is the same as of (Y_2,W_2^*) (see [2] Th. 3.1). Then, we get

$$\begin{aligned}\tilde P_1(Y_1\in\Lambda) &= \int_{\tilde\Omega_1} \underline{\delta}_T(Y_1,\tilde W_1^*)\mathbb{I}_{\{Y_1\in\Lambda\}}d\tilde{\mathbb{P}}_1^*\\ &= \int_{\tilde\Omega_2} \underline{\delta}_T(Y_2,\tilde W_2^*)\mathbb{I}_{\{Y_2\in\Lambda\}}d\tilde{\mathbb{P}}_2^* = \tilde P_2(Y_2\in\Lambda)\end{aligned}$$

for any Borelian subset Λ of $C([0,T];\mathbb{R})$; here $\mathbb{I}_.$ is the indicator function. Thus, we have uniqueness in law for equation (3.1). □

Remark 3.3 *i) The expression* (3.5) *can be written as*

$$\frac{d\mu^{a+g}}{d\mu^a}(Z) = \mathbb{E}\Big[e^{\int_0^T \alpha(s,Z)dW(s)}\big|\mathbb{F}_T(Z)\Big]e^{-\frac{1}{2}\int_0^T \alpha(s,Z)^2 ds}.$$

The same holds for other similar expressions of Radon-Nikodym derivatives appearing later on.
ii) Consider the assumptions of Theorem 3.2. Then, given a weak solution $\big(Y, (\tilde{\Omega}, \tilde{\mathbb{F}}, \{\tilde{\mathbb{F}}_t\}, \tilde{\mathbb{P}}), \tilde{W}\big)$ *of equation* (3.1)*, from* (3.8) *in the previous proof we have*

$$\frac{d\mu^a}{d\mu^{a+g}}(Y) = \tilde{\mathbb{E}}\Big[e^{-\int_0^T \alpha(s,Y)d\tilde{W}(s) - \frac{1}{2}\int_0^T \alpha(s,Y)^2 ds}\big|\mathbb{F}_T(Y)\Big] \tag{3.9}$$

$\tilde{\mathbb{P}}$*-a.s.*

4 Uniqueness in law

According to Remark 3.1, if **[A2]** holds true we set

$$\gamma(s,Y) = \sigma^+(s,Y)[b(s,Y) - a(s,Y)].$$

We have the following

Proposition 4.1 *Assume* **[A1]** *and* **[A2]**.
If there exist two weak solutions $\big(X, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ *and* $\big(X', (\Omega', \mathbb{F}', \{\mathbb{F}'_t\}, \mathbb{P}'),\ W'\big)$ *to equation* (2.1)*, with the same initial data x, such that*

$$\mathbb{P}\{\textstyle\int_0^T \gamma(s,X)^2 ds < \infty\} = \mathbb{P}'\{\int_0^T \gamma(s,X')^2 ds < \infty\} = 1, \tag{4.1}$$

then the laws of X and X' are the same.

Proof. Consider the first solution $\big(X, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$. According to **[A1]** there exists a solution Z of equation (2.2) with respect to the stochastic basis $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P})$ and the Wiener process W. For any integer $n \geq 1$, define the truncation function

$$\chi_t^n(Z) = \begin{cases} 1 & \text{if } \int_0^t \gamma(s,Z)^2 ds < n, \\ 0 & \text{otherwise.} \end{cases}$$

We have that

$$\sup_{Z\in C([0,T];\mathbb{R})} \int_0^T \chi_s^n(Z)\gamma(s,Z)^2 ds \leq n.$$

We use Theorem 3.2 with $g(s,Y) = \chi_s^n(Y)[b(s,Y) - a(s,Y)]$ in order to get that the new equation

$$Y(t) = x + \int_0^t a(s,Y)\ ds + \int_0^t \chi_s^n(Y)[b(s,Y) - a(s,Y)]\ ds + \int_0^t \sigma(s,Y)dW(s) \tag{4.2}$$

has a unique weak solution. For short, we denote its law by $\mu^{b,n}$ and we have $\mu^{b,n} \prec \mu^a$, with $\mu^{b,n}(\Lambda) = \mathbb{P}^{*n}\{Z \in \Lambda\}$, $d\mathbb{P}^{*n} = \rho^n_T(Z,W)d\mathbb{P}$, and the martingale $\rho^n = \rho^n(Z,W)$ defined by $\rho^n_t = e^{\int_0^t \chi^n_s(Z)\gamma(s,Z)dW(s) - \frac{1}{2}\int_0^t \chi^n_s(Z)\gamma(s,Z)^2 ds}$. In particular,

$$\mathbb{E}[\rho^n_t(Z,W)] = \mathbb{E}[\rho^n_0(Z,W)] = 1 \qquad \text{for all } t,$$

and

$$\frac{d\mu^{b,n}}{d\mu^a}(Z) = \mathbb{E}\big[e^{\int_0^T \chi^n_s(Z)\gamma(s,Z)dW(s) - \frac{1}{2}\int_0^T \chi^n_s(Z)\gamma(s,Z)^2 ds}|\mathbb{F}_T(Z)\big] \tag{4.3}$$

$\mathbb{P}$-a.s..
This holds for any n integer. Therefore we have uniquely defined the sequence $\{\mu^{b,n}\}_{n=1}^{\infty}$.

On the other hand, we can define a process solving equation (4.2) *with the Wiener process* W. Let us define the sequence of stopping times (depending on the process X)

$$\tau^n = \inf\{t \in [0,T] : \chi^n_t(X) = 0\} \wedge T$$

considering the infimum to be $+\infty$ when the set is empty.
Given any n, $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P})$ and W, if $\big(X, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ is a weak solution to equation (2.1) then equation

$$\begin{aligned} X^n(t) &= X(t \wedge \tau^n) \\ &\quad + \int_0^t [1 - \chi^n_s(X)]a(s, X^n)\ ds + \int_0^t [1 - \chi^n_s(X)]\sigma(s, X^n)\ dW(s) \end{aligned} \tag{4.4}$$

has a unique strong solution X^n, thanks to assumption **[A1]**. Moreover, by Itô calculus we get that this process X^n solves (4.2). Hence, $\mu^{b,n}$ coincides with the law of X^n. From (4.4) we have ($\mathbb{P}$-a.s)

$$X^n(t) = \begin{cases} X(t) & \text{on } \{\tau^n \geq t\} \\ X(\tau^n) + \int_{\tau^n}^t a(s, X^n)ds + \int_{\tau^n}^t \sigma(s, X^n)dW_s & \text{on } \{\tau^n < t\} \end{cases}$$

In particular, $X = X^n$ on the set $\{\tau^n = T\} \supseteq \{\chi^n_T(X) = 1\}$. According to (4.1) we have $\lim_{n\to\infty} \mathbb{P}\{\chi^n_T(X) = 0\} = 0$. Hence

$$\mathbb{P}\{\|X - X^n\|_{C([0,T];\mathbb{R})} > 0\} \leq \mathbb{P}\{\chi^n_T(X) = 0\} \to 0 \qquad \text{as } n \to \infty.$$

Therefore, $\mu^{b,n}$ converges to the law of X in the metric of total variation.

If we start from another solution $\big(X', (\Omega', \mathbb{F}', \{\mathbb{F}'_t\}, \mathbb{P}'),\ W'\big)$ fulfilling (4.1), we would consider the solution $\big(Z', (\Omega', \mathbb{F}', \{\mathbb{F}'_t\}, \mathbb{P}'),\ W'\big)$ to equation (2.2), giving the same $\mu^{b,n}$. Indeed, there is uniqueness in law for both equations (2.2) and (3.1). Hence, $\mu^{b,n}$ converges to the law of X'. Since the limit of $\mu^{b,n}$ is unique, we conclude that the laws of X and X' are the same. □

Remark 4.2 *Given a weak solution* $\big(X, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ *of equation* (2.1), *it is easier to construct a solution* $\big(X^n,\ (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ *of equation* (4.2) *when the noise is independent of the unknown, i.e.* $\sigma(t, X) = \sigma(t)$. *Indeed, we look for a process solving*

$$X^n(t) = x + \int_0^t a(s, X^n)\ ds + \int_0^t \chi^n_s(X^n)[b(s, X^n) - a(s, X^n)]\ ds + \int_0^t \sigma(s) dW(s).$$

Notice that, given a path X^n_ω, *this equation reduces to equation* (2.1) *if* $\chi^n(X^n_\omega) = 1$ *and to equation* (2.2) *if* $\chi^n(X^n_\omega) = 0$.

Now, we construct pathwise the solution process. For $\mathbb{P}$*-a.e.* ω, *set* $X^n_\omega(t) = X_\omega(t)$ *for* $0 \le t \le \tau^n(\omega)$. *In particular,* $\chi^n_{\tau^n(\omega)}(X^n_\omega) = 0$ *and whatever is* $X^n_\omega(t)$ *for* $t > \tau^n(\omega)$ *we will have* $\chi^n_t(X^n_\omega) = 0$ *for* $t > \tau^n(\omega)$. *Therefore the evolution of* (4.2) *on the time interval* $]\tau^n(\omega), T]$ *is given by equation* (2.2). *Summing up, we have that a solution of equation* (4.2) *with the Wiener process* W *is the process defined pathwise as follows:*

$$X^n(t) = \begin{cases} X(t), & t \in [0, \tau^n] \\ \psi^{\tau^n}(X(\tau^n), W)(t), & t \in]\tau^n, T] \end{cases}$$

where $\psi^{t_0}(y, W)$ *denotes the solution of equation* (2.2) *(with the Wiener process* W*) on the time interval* $[t_0, T]$ *and with initial data* $Z(t_0) = y$*:*

$$\psi^{t_0}(y, W)(t) = y + \int_{t_0}^t a(s, \psi^{t_0}(y, W))\ ds + \int_{t_0}^t \sigma(s)\ dW(s)$$

We point out that in this case is enough to assume that equation (2.2) *has a unique strong solution on any time interval* $[t_0, T] \subseteq [0, T]$ *and that the mapping* $[0, T] \times \mathbb{R} \times C_0([0, T]; \mathbb{R}) \ni (s, y, W) \mapsto \psi^s(y, W) \in C([s, T]; \mathbb{R})$ *is measurable. Actually for fixed initial time, this mapping is already known to have nice properties (see, e.g., [7] Ch. 4 for the properties of the mapping* $\psi : \mathbb{R} \times C_0([0, T]; \mathbb{R}) \to C([0, T]; \mathbb{R})$ *providing a strong solution on the time interval* $[0, T]$*).*

An easy example fulfilling these requirements is for the linear equation, i.e. the drift term is $a(t, Z) = c(t)Z(t)$ *with* $c, \sigma : [0, T] \to \mathbb{R}$ *measurable and bounded. Indeed, we have*

$$\psi^s(y, W)(t) = e^{\int_s^t c(u) du} y + \int_s^t e^{\int_u^t c(r) dr} \sigma(u) dW(u), \qquad t \in [s, T]. \tag{4.5}$$

5 Absolute continuity of μ^b with respect to μ^a

We consider equations (2.1) and (2.2) with the same initial data $x \in \mathbb{R}$. We have the following result. The assumptions are the same as for the uniqueness result of the previous section; therefore we denote by μ^b the unique law for equation (2.1).

Let us denote by $\chi_t(Z)$ the indicator function of the set $\{\int_0^t \gamma(s, Z)^2 ds < \infty\}$.

Proposition 5.1 *Assume* **[A1]** *and* **[A2]**.
If there exists a weak solution $\big(X,\ (\Omega,\mathbb{F},\{\mathbb{F}_t\},\mathbb{P}),\ W\big)$ *to equation* (2.1) *such that*

$$\mathbb{P}\{\int_0^T \gamma(s,X)^2 ds < \infty\} = 1, \tag{4.1$'$}$$

then $\mu^b \prec \mu^a$. *Moreover,*

$$\frac{d\mu^b}{d\mu^a}(Z) = \mathbb{E}\big[e^{\mathcal{I}_T(Z) - \frac{1}{2}\int_0^T \gamma(s,Z)^2 ds}|\mathbb{F}_T(Z)\big] \qquad \mathbb{P}-a.s., \tag{5.1}$$

where

$$\mathcal{I}_T(Z) = \mathbb{P} - \lim_{n\to\infty} \chi_T(Z)\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s). \tag{5.2}$$

Proof. Going back to the proof of Proposition 4.1, we have that $\mu^{b,n} \prec \mu^a$ and $\|\mu^{b,n} - \mu^b\|_{var} \to 0$. Then, if $\mu^a(\Lambda) = 0$ for some Borelian subset Λ of $C([0,T];\mathbb{R})$, then $\mu^{b,n}(\Lambda) = 0$ and finally $\mu^b(\Lambda) = 0$. This proves $\mu^b \prec \mu^a$.

Moreover, $\|\mu^{b,n} - \mu^b\|_{var} \to 0$ implies that $\mu^{b,n}$ (equivalently, $\mathbb{P}^{*n}$) is a Cauchy sequence in the metric of total variation. Since $\|\mathbb{P}^{*n} - \mathbb{P}^{*m}\|_{var} = \|\frac{d\mathbb{P}^{*n}}{d\mathbb{P}} - \frac{d\mathbb{P}^{*m}}{d\mathbb{P}}\|_{L^1(\mathbb{P})}$, this is the same as saying that

$$\frac{d\mathbb{P}^{*n}}{d\mathbb{P}} = \rho_T^n(Z,W) = e^{\int_0^t \chi_s^n(Z)\gamma(s,Z)dW(s) - \frac{1}{2}\int_0^t \chi_s^n(Z)\gamma(s,Z)^2 ds}$$

is a Cauchy sequence in the metric of $L^1(\mathbb{P})$. Therefore $\rho_T^n(Z,W)$ converges in the norm of $L^1(\mathbb{P})$ to some limit, which is denoted by $\rho_T(Z,W)$. We want to identify $\rho_T(Z,W)$.

Notice that if $\int_0^T \gamma(s,Z)^2 ds < \infty$ $\mathbb{P}$-a.s., then the stochastic integral in the exponent of $\rho_T^n(Z,W)$ would converge in probability to $\int_0^T \gamma(s,Z)dW(s)$ (see [11], Section 4.2.6) and the deterministic integral to $\int_0^T \gamma(s,Z)^2 ds$. Otherwise, we proceed following the argument given in [11] (Section 4.2.9), but with some modification. The random variable

$$\mathcal{I}_T^n(Z) := \chi_T(Z)\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s)$$

converges in probability. Indeed, $\chi_T(Z)\int_0^T |\chi_s^n(Z)\gamma(s,Z) - \gamma(s,Z)|^2 ds$ converges to 0 $\mathbb{P}$-a.s.; hence there is convergence in probability. Therefore, by Lemma 4.6 of [11], $\mathcal{I}_T^n(Z)$ $(n = 1,2,\dots)$ is a Cauchy sequence in probability. It follows that it converges in probability to a random variabile, which we denote by $\mathcal{I}_T(Z)$.

First, we have

$$\begin{aligned}
\chi_T(Z)\rho_T &= \chi_T(Z)\ \mathbb{P} - \lim_n e^{\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s) - \frac{1}{2}\int_0^T \chi_s^n(Z)\gamma(s,Z)^2 ds} \\
&= \mathbb{P} - \lim_n \chi_T(Z)e^{\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s) - \frac{1}{2}\int_0^T \chi_s^n(Z)\gamma(s,Z)^2 ds} \\
&= \chi_T(Z)e^{\mathbb{P}-\lim_n[\chi_T(Z)\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s) - \frac{1}{2}\chi_T(Z)\int_0^T \chi_s^n(Z)\gamma(s,Z)^2 ds]} \\
&= \chi_T(Z)e^{\mathcal{I}_T(Z) - \frac{1}{2}\int_0^T \gamma(s,Z)^2 ds} \qquad \mathbb{P}-a.s.
\end{aligned}$$

This means that

$$\rho_T(Z,W) = e^{\mathcal{I}_T(Z) - \frac{1}{2}\int_0^T \gamma(s,Z)^2 ds} \tag{5.3}$$

a.s. on the set $\{\chi_T(Z) = 1\}$.

Now let us check that (5.3) holds a.s. also on the set $\{\chi_T(Z) = 0\}$, or equivalently a.s. on the set $\{\int_0^T \gamma(s,Z)^2 ds = \infty\}$. We analyze the left and right hand side of equality (5.3). The r.h.s. of (5.3) vanishes a.s. on the set $\{\int_0^T \gamma(s,Z)^2 ds = \infty\}$. Indeed, by definition $\mathcal{I}_T^n(Z) = 0$ a.s. on $\{\chi_T(Z) = 0\}$ and therefore $\mathcal{I}_T(Z) = 0$ a.s. on $\{\chi_T(Z) = 0\}$.
On the other hand, the l.h.s. of (5.3) vanishes a.s. on the set $\{\int_0^T \gamma(s,Z)^2 ds = \infty\}$. Indeed, on this set

$$\rho_T^n(Z,W) = e^{\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s) - \frac{1}{2}n}, \qquad \mathbb{P}-a.s.$$

Therefore

$$\begin{aligned} &\rho_T^n(Z,W) \le e^{-\frac{n}{4}} \qquad \mathbb{P}-a.s. \\ &\qquad \text{on } \{\textstyle\int_0^T \gamma(s,Z)^2 ds = \infty\} \cap \{\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s) \le \frac{n}{4}\}. \end{aligned} \tag{5.4}$$

Using Chebyshev inequality we get

$$\begin{aligned} \mathbb{P}\{\textstyle\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s) > \frac{n}{4}\} &= \frac{1}{2}\mathbb{P}\{|\textstyle\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s)| > \frac{n}{4}\} \\ &\le \frac{1}{2}\frac{\mathbb{E}[\int_0^T \chi_s^n(Z)\gamma(s,Z)^2 ds]}{(n/4)^2} \\ &\le \frac{1}{2}\frac{n}{(n/4)^2} \longrightarrow 0 \ \text{ as } n \to \infty. \end{aligned} \tag{5.5}$$

Let $\chi^{W,n}$ be the indicator function of the set $\{\int_0^T \chi_s^n(Z)\gamma(s,Z)dW(s) \le \frac{n}{4}\}$. According to (5.5) we have that

$$\lim_{n\to\infty} \mathbb{P}\{\chi^{W,n} = 0\} = 0. \tag{5.6}$$

We investigate the convergence of $\rho_T^n(Z,W)$ on the set $\{\chi_T(Z) = 0\}$: for any $\varepsilon > 0$ we have

$$\begin{aligned} &\mathbb{P}\{\rho_T^n(Z,W)[1-\chi_T(Z)] > \varepsilon\} \\ &\ = \mathbb{P}\{\rho_T^n(Z,W) > \varepsilon,\ \chi_T(Z) = 0\} \\ &\ = \mathbb{P}\{\rho_T^n(Z,W) > \varepsilon,\ \chi_T(Z) = 0,\ \chi^{W,n} = 0\} + \mathbb{P}\{\rho_T^n(Z,W) > \varepsilon,\ \chi_T(Z) = 0,\ \chi^{W,n} = 1\} \\ &\ \le \mathbb{P}\{\chi^{W,n} = 0\} + \mathbb{P}\{e^{-\frac{n}{4}} > \varepsilon,\ \chi_T(Z) = 0,\ \chi^{W,n} = 1\} \quad \text{by (5.4)} \\ &\ \le \mathbb{P}\{\chi^{W,n} = 0\} + \mathbb{P}\{e^{-\frac{n}{4}} > \varepsilon\} \longrightarrow 0 \quad \text{ as } n \to \infty \text{ by (5.6)}. \end{aligned}$$

This implies that $\rho_T(Z,W)[1-\chi_T(Z)] = 0$ a.s.; hence, $\rho_T(Z,W) = 0$ a.s. on the set $\{\chi_T(Z) = 0\}$.

We conclude that

$$\rho_T(Z,W) = e^{\mathcal{I}_T(Z) - \frac{1}{2}\int_0^T \gamma(s,Z)^2 ds} \qquad \mathbb{P}- \text{ a.s.} \tag{5.7}$$

Finally, denoting by $\mathbb{P}^*$ the limit of $\mathbb{P}^{*n}$, so that $\mu^b(\Lambda) = \mathbb{P}^*\{Z \in \Lambda\}$, we have proved that $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \rho_T(Z,W)$. As done in the proof of Theorem 3.2, we get (5.1). □

6 Equivalence of the laws

As noticed in the previous section, if $\mathbb{P}\{\int_0^T \gamma(s,Z)^2 ds < \infty\} = 1$, then

$$\mathcal{I}_T(Z) = \int_0^T \gamma(s,Z) dW(s) \tag{6.1}$$

and therefore $\dfrac{d\mathbb{P}^*}{d\mathbb{P}} = \rho_T(Z,W)$, where

$$\rho_t(Z,W) = e^{\int_0^t \gamma(s,Z)dW(s) - \frac{1}{2}\int_0^t \gamma(s,Z)^2 ds}$$

is a strictly positive martingale.

From this, we have a result on how to use Girsanov transform under very weak assumptions (basically, avoiding Novikov condition or similar conditions involving the expectation of the exponential of a random variable related to the integral of $\gamma(s,Z)$; see [13], [9], [10]).

Theorem 6.1 *Assume* **[A1]**, **[A2]** *and that equation* (2.1) *has a weak solution* $\big(X,\ (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$. *Denote by Z the unique solution of equation* (2.2) *with respect to the same stochastic basis and Wiener process.*
If

$$\mathbb{P}\{\textstyle\int_0^T \gamma(s,X)^2 ds < \infty\} = 1, \tag{6.2}$$

$$\mathbb{P}\{\textstyle\int_0^T \gamma(s,Z)^2 ds < \infty\} = 1, \tag{6.3}$$

then
i) the process $\rho = \rho(Z,W)$ given by

$$\rho_t = e^{\int_0^t \gamma(s,Z)dW(s) - \frac{1}{2}\int_0^t \gamma(s,Z)^2 ds}, \quad 0 \le t \le T, \tag{6.4}$$

is a positive $\{\mathbb{F}_t\}$-martingale; in particular

$$\mathbb{E}[\rho_t(Z,W)] = 1 \ \textit{ for any } t \in [0,T]. \tag{6.5}$$

ii)

$$W^*(t) = W(t) - \int_0^t \gamma(s,Z)\ ds\ , \qquad t \in [0,T], \tag{6.6}$$

is a Wiener process with respect to $\mathbb{P}^*$, where the probability measure $\mathbb{P}^*$ is defined on $(\Omega, \mathbb{F}_T)$ by

$$d\mathbb{P}^* = \rho_T(Z,W)\ d\mathbb{P}. \tag{6.7}$$

Proof. *i*) Notice that the exponential process $\rho(Z,W)$ is a positive local martingale and then, by Fatou lemma, a supermartingale. Since $\rho_0(Z,W)=1$, it is enough to have $\mathbb{E}[\rho_T(Z,W)]=1$ in order to prove that it is a martingale. But, $\rho_T(Z,W)$ is the $L^1(\mathbb{P})$-limit of $\rho_T^n(Z,W)$; since we already know from the proof of Proposition 4.1 that

$$\mathbb{E}[\rho_T^n(Z,W)]=1 \qquad \text{for any } n=1,2,\dots$$

we get that $\mathbb{E}[\rho_T(Z,W)]=1$.

ii) Given *i*), this is Girsanov theorem (see, e.g., [6]). □

Now we state our main result.

Theorem 6.2 *Assume* **[A1]**, **[A2]** *and that equation* (2.1) *has a weak solution* $\big(X,\ (\Omega,\mathbb{F},\{\mathbb{F}_t\},\mathbb{P}),\ W\big)$. *Denote by* Z *the unique solution of equation* (2.2) *with respect to the same stochastic basis and Wiener process.*
If (6.2)-(6.3) *hold, then the law of the solution of equation* (2.1) *is unique. Moreover,* $\mu^b \sim \mu^a$. *In particular,*

$$\frac{d\mu^b}{d\mu^a}(Z)=\mathbb{E}\left[e^{+\int_0^T\gamma(s,Z)dW(s)-\frac{1}{2}\int_0^T\gamma(s,Z)^2ds}\big|\mathbb{F}_T(Z)\right] \qquad \mathbb{P}-a.s. \tag{6.8}$$

$$\frac{d\mu^a}{d\mu^b}(Z)=\mathbb{E}\left[e^{-\int_0^T\gamma(s,Z)dW(s)+\frac{1}{2}\int_0^T\gamma(s,Z)^2ds}|\mathbb{F}_T(Z)\right] \qquad \mathbb{P}-a.s. \tag{6.9}$$

$$\frac{d\mu^a}{d\mu^b}(Z)=\mathbb{E}^*\left[e^{-\int_0^T\gamma(s,Z)dW^*(s)-\frac{1}{2}\int_0^T\gamma(s,Z)^2ds}|\mathbb{F}_T(Z)\right] \qquad \mathbb{P}^*-a.s. \tag{6.10}$$

$$\frac{d\mu^a}{d\mu^b}(X)=\mathbb{E}\left[e^{-\int_0^T\gamma(s,X)dW(s)-\frac{1}{2}\int_0^T\gamma(s,X)^2ds}\big|\mathbb{F}_T(X)\right] \qquad \mathbb{P}-a.s. \tag{6.11}$$

where $\mathbb{P}^*, W^*$ *are defined by* (6.7), (6.6) *respectively.*

Proof. Uniqueness in law comes from Proposition 4.1, $\mu_b \prec \mu_a$ from Proposition 5.1 and (6.8) from (5.1), (6.1) with the assumption (6.3).

Moreover, (6.3) implies that $\mathbb{P}\{\rho_T(Z,W)=0\}=0$. Then $\mathbb{P}\prec\mathbb{P}^*$ with

$$\frac{d\mathbb{P}}{d\mathbb{P}^*}=\frac{1}{\rho_T(Z,W)}, \qquad \mathbb{P}^*-a.s.$$

(see Lemma 6.8 in [11]). From $\mathbb{P}\prec\mathbb{P}^*$ follows $\mu^a\prec\mu^b$.

As done in the proof of Theorem 3.2, from $\frac{d\mathbb{P}}{d\mathbb{P}^*}=\big(\rho_T(Z,W)\big)^{-1}$ we get (6.9). Moreover, using (6.6) we get

$$\frac{d\mathbb{P}}{d\mathbb{P}^*}=e^{-\int_0^T\gamma(s,Z)dW(s)+\frac{1}{2}\int_0^T\gamma(s,Z)^2ds}=e^{-\int_0^T\gamma(s,Z)dW^*(s)-\frac{1}{2}\int_0^T\gamma(s,Z)^2ds};$$

in the same way, this gives (6.10). In particular,

$$\mathbb{E}^*\Big[e^{-\int_0^T\gamma(s,Z)dW^*(s)-\frac{1}{2}\int_0^T\gamma(s,Z)^2ds}\Big]=1. \tag{6.12}$$

This is written for the solution $\big(Z,\ (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}^*),\ W^*\big)$ of equation (2.1). Since there is uniqueness in law for equation (2.1), if we consider the same relationship for the solution $\big(X,\ (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ we get

$$\mathbb{E}\Big[e^{-\int_0^T \gamma(s,X)dW(s) - \frac{1}{2}\int_0^T \gamma(s,X)^2 ds}\Big] = 1. \tag{6.13}$$

Now, let us start from equation (2.1) and consider equation (2.2) as a modification of equation (2.1) by a change of drift. Thanks to (6.13), we can use Girsanov theorem. Similarly to what we have done in Theorem 3.2 and Remark 3.3 ii), we get (6.11). □

7 Conclusions

We compare our results with whose of Liptser and Shiryaev. Let us begin reminding a result of [11]: there Theorem 7.18 states that if we assume **[A1]**, **[A2]**, that equation (2.1) has a weak solution $\big(X,\ (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ satisfying (6.2) and

$$\mathbb{E}\Big[e^{-\int_0^T \gamma(s,X)dW(s) - \frac{1}{2}\int_0^T \gamma(s,X)^2 ds}\Big] = 1, \tag{7.1}$$

then $\mu^b \sim \mu^a$ and

$$\frac{d\mu^a}{d\mu^b}(X) = \mathbb{E}\Big[e^{-\int_0^T \gamma(s,X)dW(s) - \frac{1}{2}\int_0^T \gamma(s,X)^2 ds}\big|\mathbb{F}_T(X)\Big], \qquad \mathbb{P}-a.s.$$

The crucial issue is how to get (7.1) without assuming the quite strong Novikov condition (see [13])

$$\mathbb{E}\Big[e^{\frac{1}{2}\int_0^T \gamma(s,X)^2 ds}\Big] < \infty$$

or other conditions involving the expectation of the exponential of a random variable related to the integral of $\gamma(s,X)$ (see [9], [10]). This is done in our Theorem 6.2 with the "$\mathbb{P}$-a.s." conditions (6.2)-(6.3).

However, Liptser and Shiryaev present another result, more operative than Theorem 7.18. This is Theorem 7.19 of [11] providing $\mu^b \sim \mu^a$ with the same assumptions of Theorem 7.18 except (7.1), which is replaced by

$$\mathbb{P}\{\int_0^T \gamma_a(s,X)^2 ds < \infty\} = \mathbb{P}\{\int_0^T \gamma_b(s,X)^2 ds < \infty\} = 1, \tag{7.2}$$

$$\mathbb{P}\{\int_0^T \gamma_a(s,Z)^2 ds < \infty\} = \mathbb{P}\{\int_0^T \gamma_b(s,Z)^2 ds < \infty\} = 1, \tag{7.3}$$

where

$$\gamma_a(s,X) = \sigma^+(s,X)a(s,X), \qquad \gamma_b(s,X) = \sigma^+(s,X)b(s,X).$$

Because of $\gamma = \gamma_b - \gamma_a$, assumptions (7.2)-(7.3) are stronger than (6.2)-(6.3).

Therefore, we can see our Theorem 6.2 as an intermediate result between the two theorems of [11]. We have the same result as Theorem 7.18, but saying

concretely how to get (7.1) with "$\mathbb{P}$-a.s." conditions. This is in the same spirit as Theorem 7.19. However, our conditions (6.2)-(6.3) involve only the difference $b-a$ of the drift terms, whereas conditions (7.2)-(7.3) involve both the drift terms b and a.

We point out that our results on the absolute continuity of the laws are identical to [11], but the expressions of the Radon-Nikodym derivatives are different from those of Liptser and Shiryaev. In fact, under (7.2)-(7.3) Theorem 7.19 gives

$$\frac{d\mu^b}{d\mu^a}(Z) = e^{\int_0^T \sigma^+(s,Z)^2[b(s,Z)-a(s,Z)]\ dZ(s) - \frac{1}{2}\int_0^T \sigma^+(s,Z)^2[b(s,Z)^2-a(s,Z)^2]\ ds} \tag{7.4}$$

and

$$\frac{d\mu^a}{d\mu^b}(X) = e^{-\int_0^T \sigma^+(s,X)^2[b(s,X)-a(s,X)]dX(s) + \frac{1}{2}\int_0^T \sigma^+(s,X)^2[b(s,X)^2-a(s,X)^2]ds} \tag{7.5}$$

($\mathbb{P}$-a.s.). Let us show that (7.4) can be obtained from (6.8); similarly, for (7.5) from (6.11). If this is true, then we conclude that with our proofs we get the same result as Liptser and Shiryaev. However, our proofs are different from [11] basically in one point: Liptser and Shiryaev analyze the equation satisfied by the Radon-Nikodym derivative, whereas we analyze the Radon-Nikodym derivative as the limit of the sequence $\frac{d\mathbb{P}^{*n}}{d\mathbb{P}}$. This makes our proofs shorter.

Let us come back to the expression of the Radon-Nikodym derivative $\frac{d\mu^b}{d\mu^a}(Z)$. Now, we assume (7.2)-(7.3); of course our results hold true. Therefore $\frac{d\mu^b}{d\mu^a}(Z)$ is given by (6.8). From (2.2) we have W depending on Z: $dW(t) = \sigma^+(t,Z)[dZ(t) - a(t,Z)\ dt]$. Then

$$\int_0^T \gamma(s,Z)dW(s) - \frac{1}{2}\int_0^T \gamma(s,Z)^2 ds$$

becomes (formally)

$$\int_0^T \sigma^+(s,Z)^2[b(s,Z)-a(s,Z)]dZ(s) - \frac{1}{2}\int_0^T \sigma^+(s,Z)^2[b(s,Z)^2-a(s,Z)^2]ds. \tag{7.6}$$

Since

$$\int_0^T |\sigma^+(s,Z)^2[b(s,Z)-a(s,Z)]|^2\sigma(s,Z)^2\ ds \le \int_0^T \gamma(s,Z)^2\ ds,$$

the stochastic integral in (7.6) is well defined if (6.3) holds, whereas the deterministic integral requires (7.3). Then, (7.6) is in fact well defined with assumption (7.3) and it depends only on Z and not also on W. From (6.8) and (7.6), we get (7.4).

8 Bigger dimensions

Let $d, m \in \mathbb{N}$ with $dm > 1$. The solution processes X and Z have paths in $C([0,T];\mathbb{R}^d)$, the initial data $x \in \mathbb{R}^d$ and W is an m-dimensional Wiener process. Any vector v is a column vector, whose transposed is the row vector v^T. We set $\|X\|^2 = \sum_{i=1}^d X_i^2$.

Let $\mathcal{B}_t$ be the σ-algebra of Borelian subsets of $C([0,t];\mathbb{R}^d)$, for $0 < t \le T$. The drift terms a and b are $\mathbb{R}^d$-valued non anticipative measurable functionals, that is

$$a, b : [0,T] \times C([0,T];\mathbb{R}^d) \to \mathbb{R}^d,$$

are measurable and, for each $t \in [0,T]$, $a(t,\cdot), b(t,\cdot)$ are $\mathcal{B}_t$-measurable. Similarly, the diffusion term $\sigma : [0,T] \times C([0,T];\mathbb{R}^d) \to \mathbb{R}^d \times \mathbb{R}^m$ is a non anticipative measurable functional; in particular $(\sigma W)_i = \sum_{k=1}^m \sigma_{ik} W_k$ for $i = 1, \ldots, d$. The entries satisfy the previous conditions; the two main assumptions become

$$[\mathbf{A1}] \left[\begin{array}{l} \exists \text{ constants } L_1, L_2 \text{ and a function } K \text{ non decreasing and right continuous,} \\ \text{with } 0 \le K(s) \le 1, \text{ such that all the components } a_i, b_{ik} \text{ satisfy} \\ a_i(t,Y)^2 + \sigma_{ik}(t,Y)^2 \le L_1 \int_0^t [1 + \|Y(s)\|^2] dK(s) + L_2[1 + \|Y(t)\|^2] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t \in [0,T], Y \in C([0,T];\mathbb{R}^d) \\ \text{and} \\ |a_i(t,Y_1) - a_i(t,Y_2)|^2 + |\sigma_{ik}(t,Y_1) - \sigma_{ik}(t,Y_2)|^2 \\ \qquad\qquad \le L_1 \int_0^t \|Y_1(s) - Y_2(s)\|^2 dK(s) + L_2 \|Y_1(t) - Y_2(t)\|^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t \in [0,T], Y_1, Y_2 \in C([0,T];\mathbb{R}^d) \end{array} \right.$$

$$[\mathbf{A2}] \left[\begin{array}{l} \exists \gamma \text{ finite and } \mathbb{R}^m\text{-valued non anticipative measurable functional:} \\ \quad \sigma(s,Y)\gamma(s,Y) = b(s,Y) - a(s,Y) \qquad \forall s \in [0,T], Y \in C([0,T];\mathbb{R}^d). \end{array} \right.$$

In Remark 4.2, the solution mapping is

$$[0,T] \times \mathbb{R}^d \times C_0([0,T];\mathbb{R}^m) \ni (s,y,W) \mapsto \psi^s(y,W) \in C([s,T];\mathbb{R}^d)$$

and the linear equation in the example has solution still given by (4.5), where $c : [0,T] \to \mathbb{R}^d \times \mathbb{R}^d$ is measurable and bounded.

All the results of the previous sections hold true with the suitable change of notations. Mainly, γ^2 becomes $\|\gamma\|^2 = \sum_{k=1}^m \gamma_k^2$ and σ^+ is the pseudo-inverse matrix of σ (see, e.g., [1], [12]); σ^+ is an $m \times d$-matrix, uniquely defined.

However, let us investigate this multidimensional problem in details. Assumption **[A2]** refers to the linear system of d equations in m unknowns

$$\sigma(s,Y)\gamma(s,Y) = b(s,Y) - a(s,Y) \tag{8.1}$$

and is a consistency condition involving σ and $b - a$ (see, e.g., [1] for all the results on linear systems and matrices). Moreover, if a solution γ exists and $\operatorname{rank}\sigma = m$ then the solution of (8.1) is unique and is given by

$$\gamma(s,Y) = \sigma^+(s,Y)[b(s,Y) - a(s,Y)]. \tag{8.2}$$

In particular, if σ has maximal rank we have

$$\sigma^+ = \sigma^T(\sigma\sigma^T)^{-1} \qquad \text{or} \;\; \sigma^+ = (\sigma^T\sigma)^{-1}\sigma^T. \tag{8.3}$$

(Notice that if the rank of σ is maximal, then also the square matrices $\sigma\sigma^T$ and $\sigma^T\sigma$ have maximal rank and therefore are invertible.) And if the matrix σ vanishes, then also σ^+ has all the entries equal to 0.
Otherwise, there are infinite solutions of (8.1), one of them being given by (8.2). This is the case of dimensions $d \geq m$ with rank of σ not maximal ($< m$), or of dimensions $m > d$. To handle these cases, let us recall the singular value decomposition of the $d \times m$-matrix σ with $\operatorname{rank}\sigma = r$, $r \leq \min(d, m)$:

$$\sigma = \lambda^{(1)}u^{(1)}(v^{(1)})^T + \lambda^{(2)}u^{(2)}(v^{(2)})^T + \ldots + \lambda^{(r)}u^{(r)}(v^{(r)})^T,$$

where $\lambda^{(1)} \geq \lambda^{(2)} \geq \ldots \lambda^{(r)} > 0$, $\{u^{(i)}\}_{i=1}^r$ is an orthonormal set of d-dimensional vectors and $\{v^{(i)}\}_{i=1}^r$ is an orthonormal set of m-dimensional vectors. Moreover

$$\sigma^+ = \frac{1}{\lambda^{(1)}}v^{(1)}(u^{(1)})^T + \frac{1}{\lambda^{(2)}}v^{(2)}(u^{(2)})^T + \ldots + \frac{1}{\lambda^{(r)}}v^{(r)}(u^{(r)})^T.$$

Then the d-dimensional vector $\sigma(t, X)dW(t)$ can be written as

$$\lambda^{(1)}(t,X)u^{(1)}(t,X)v^{(1)}(t,X)^T dW(t) + \ldots + \lambda^{(r)}(t,X)u^{(r)}(t,X)v^{(r)}(t,X)^T dW(t).$$

This means that

$$\sigma(t,X)dW(t) = \tilde{\sigma}(t,X)d\tilde{W}^X(t)$$

with $\tilde{\sigma}(t, X)$ the $d \times r$-matrix and $\tilde{W}^X(t)$ the r-vector defined by

$$\begin{aligned}
&\tilde{\sigma}_{ij}(t,X) = \lambda^{(j)}(t,X)u_i^{(j)}(t,X), \\
&\tilde{W}_i^X(t) = \int_0^t v^{(i)}(s,X)^T dW(s) \equiv \sum_{k=1}^m \int_0^t v_k^{(i)}(s,X)dW_k(s).
\end{aligned}$$

The matrix $\tilde{\sigma}(t, X)$ has maximal rank ($= r$), since the vectors $u^{(j)}$ are orthogonal to each other. Moreover, $\tilde{W}^X$ is an r-dimensional Wiener process. Indeed, the components of this vectors are one dimensional independent Wiener processes, thanks to the fact that $\{v^{(i)}(s, X)\}_{i=1}^r$ is an orthonormal set of m-dimensional vectors.

Now, let us read equation (2.1) with $\tilde{\sigma}(t, X)d\tilde{W}^X(t)$ instead of $\sigma(t, X)dW(t)$; similarly for equation (2.2). Then, according to the previous considerations for the $d \times r$-diffusion matrix $\tilde{\sigma}$ with $r < d$ and maximal rank ($= r$), we get that the system

$$\tilde{\sigma}(s,Y)\tilde{\gamma}(s,Y) = b(s,Y) - a(s,Y);$$

has at most one solution, given by

$$\tilde{\gamma}(s,Y) = \tilde{\sigma}^+(s,Y)[b(s,Y) - a(s,Y)]$$

with $\tilde{\sigma}^+ = (\tilde{\sigma}^T\tilde{\sigma})^{-1}\tilde{\sigma}^T$. Actually, $\tilde{\sigma}$ has r columns given by the vectors $\lambda^{(i)}u^{(i)}$ and $\tilde{\sigma}^+$ has r rows given by the vectors $(\frac{1}{\lambda^{(i)}}u^{(i)})^T$.

A tedious but easy computation provides

$$\|\sigma^+(b-a)\|^2 = \sum_{i=1}^{r} \left| \frac{{u^{(i)}}^T(b-a)}{\lambda^{(i)}} \right|^2 = \|\tilde{\sigma}^+(b-a)\|^2.$$

Since the latter quantity is uniquely defined, also the first is unique. Therefore

$$\int_0^T \|\gamma(s,Y)\|^2 ds = \int_0^T \|\tilde{\gamma}(s,Y)\|^2 ds \quad \text{ for } Y \in C([0,T];\mathbb{R}^d),$$

where

$$\gamma(s,Y) = \sigma^+(s,Y)[b(s,Y) - a(s,Y)].$$

This expression of γ provides the unique relevant solution of (8.1) in the Girsanov transform, even when the solution of (8.1) is not unique. In particular, we have

$$\frac{d\mu^b}{d\mu^a}(Z) = \mathbb{E}\Big[e^{+\int_0^T \gamma(s,Z)dW(s) - \frac{1}{2}\int_0^T \|\gamma(s,Z)\|^2 ds} \Big| \mathbb{F}_T(Z)\Big],$$

$$\frac{d\mu^a}{d\mu^b}(X) = \mathbb{E}\Big[e^{-\int_0^T \gamma(s,X)dW(s) - \frac{1}{2}\int_0^T \|\gamma(s,X)\|^2 ds} \Big| \mathbb{F}_T(X)\Big],$$

a.s., when we assume

$$\mathbb{P}\{\textstyle\int_0^T \|\gamma(s,X)\|^2 ds < \infty\} = 1, \quad \mathbb{P}\{\textstyle\int_0^T \|\gamma(s,Z)\|^2 ds < \infty\} = 1,$$

instead of (6.2)-(6.3).

We therefore conclude that we get all our previous results, included the uniqueness result. Let us emphasize that the uniqueness question is not investigated in [11] for $dm > 1$ when equation (8.1) has more than one solution. However, even if not stated, it appears clear from the results of [11] in the one dimensional case that there is uniqueness in law for equation (2.1), because of the uniqueness of γ (see also the beginning of Section 4).

9 Applications

Let us consider the case of $b = a + f$, that is we deal with

$$\begin{aligned} dX(t) &= a(t,X)\ dt + f(t,X)\ dt + \sigma(t,X)\ dW(t), \qquad && X(0) = x \\ dZ(t) &= a(t,Z)\ dt + \sigma(t,Z)\ dW(t), && Z(0) = x \end{aligned}$$

To apply the results of [11], besides **[A1]** and **[A2]** we have to check conditions on a and $a + f$, whereas our results require only a condition on f. Let us see how to use our results; first, in the one dimensional problem, then in the infinite dimensional one.

One dimensional stochastic differential equations

We consider conditions involving the process X; of course, the same holds true for those involving Z.

Our condition (6.2) becomes

$$\mathbb{P}\{\int_0^T \sigma^+(t,X)^2 f(t,X)^2\ dt < \infty\} = 1, \tag{9.1}$$

whereas (7.2) becomes

$$\mathbb{P}\{\int_0^T \sigma^+(t,X)^2 \big(a(t,X)^2 + [a(t,X)+f(t,X)]^2\big)\ dt < \infty\} = 1, \tag{9.2}$$

that is

$$\begin{aligned}&\mathbb{P}\{\int_0^T \sigma^+(t,X)^2 a(t,X)^2\ dt < \infty\}\\ &\qquad\qquad = 1 = \mathbb{P}\{\int_0^T \sigma^+(t,X)^2 [2a(t,X)f(t,X) + f(t,X)^2]\ dt < \infty\}.\end{aligned}$$

If $af \geq 0$ this is equivalent to

$$\mathbb{P}\{\int_0^T \sigma^+(t,X)^2 a(t,X)^2\ dt < \infty\} = \mathbb{P}\{\int_0^T \sigma^+(t,X)^2 f(t,X)^2\ dt < \infty\} = 1, \tag{9.3}$$

In general, the latter implies (9.2).

This condition (9.3) is stronger than (9.1), *unless* σ is constant. In fact, if σ is a constant $\neq 0$, then $\mathbb{P}\{\int_0^T \sigma^+(t,X)^2 a(t,X)^2\ dt < \infty\} = 1$ becomes

$$\mathbb{P}\{\int_0^T a(t,X)^2\ dt < \infty\} = 1 \tag{9.4}$$

which is trivially fulfilled thanks to the growth condition on a included in **[A1]**. Then we only have to check if

$$\mathbb{P}\{\int_0^T f(t,X)^2\ dt < \infty\} = 1,$$

that is (9.1) and (9.3) are equivalent.

Otherwise, for general σ, condition (9.3) is stronger than our condition (9.1).

Infinite dimensional stochastic differential equations

To have weaker assumption is even more important in the infinite dimensional setting. In fact, *even if* σ is constant, the conditions of Liptser and Shiryaev

$$\begin{aligned}\mathbb{P}\{\int_0^T \|\sigma^+(s,X)a(s,X)\|^2\ ds < \infty\} = 1\\ \mathbb{P}\{\int_0^T \|\sigma^+(s,Z)a(s,Z)\|^2\ ds < \infty\} = 1\end{aligned} \tag{9.5}$$

may be cumbersome (see next Remark 9.3). This is different from the finite dimensional framework. Indeed, the coefficients σ and a are now operators in some infinite dimensional spaces. Our results allow to obtain uniqueness in law and absolute continuity of the laws getting rid of (9.5).

First, we fix the Hilbert spaces to work in and we make precise the norm to consider in (9.5). We are given separable Hilbert spaces $E \subseteq E_1 \subseteq H$ with continuous and dense embeddings. The space E will "replace" the state space $\mathbb{R}^d$. We denote by $\|\cdot\|_H$ the norm in H and by ${}_H\langle\cdot,\cdot\rangle_H$ the scalar product in H.

For simplicity, let us consider the very simple but interesting case of constant diffusion and drift independent of the first variable t and linear in the second variable X. Equation (2.2) becomes

$$dZ(t) = AZ(t)\;dt + \sqrt{Q}\;dW(t), \qquad Z(0) = x \tag{9.6}$$

where W is a cylindrical Wiener process in H, defined on the probability space $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P})$. This means that, if $\{e_j\}_{j=1}^\infty$ is a complete orthonormal system of H, then we represent $W(t) = \sum_j \beta_j(t)e_j$ with $\{\beta_j\}_{j=1}^\infty$ a sequence of i.i.d. one dimensional Wiener processes defined on $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P})$, . The operators A and Q are linear operators in H and $x \in E$. Therefore equation (9.6) is a linear stochastic equation; this is the simplest infinite dimensional equation to deal with, for which it is easy to get existence and uniqueness of solutions and of invariant measures (see, e.g., [3], [4]). More general equations can be deal with in a similar way; but (9.6) allows us already to cover interesting examples.

Instead of (2.1), consider the semilinear stochastic equation

$$dX(t) = [AX(t) + F(X(t))]\;dt + \sqrt{Q}dW(t), \qquad X(0) = x \tag{9.7}$$

where $F : E \to E_1$ is measurable.

According to Remark 4.2, we assume

$$[\mathbf{A3}] \left[\begin{array}{l} \text{for any initial data } x \in E \text{ and on any time interval } [t_0, T] \subseteq [0, T] \\ \text{equation (9.6) has a unique strong solution } Z, \text{ whose paths are in} \\ C([t_0, T]; E) \text{ a.s.} \end{array}\right.$$

General conditions on A and Q to get it, can be found, e.g., in [3], whereas examples are in [3] and [5].

Moreover

$$[\mathbf{A4}] \left[\begin{array}{l} Ran(F) \subseteq Ran(\sqrt{Q}) \\ \exists\ (\sqrt{Q})^{-1} \end{array}\right.$$

We set

$$\Gamma(Y) = (\sqrt{Q})^{-1}F(Y) \qquad \forall Y \in E.$$

We have that $\Gamma : E \to H$ is measurable.

Here is our result of uniqueness in law of Section 4, stated in the infinite dimensional setting.

Proposition 9.1 *Assume* **[A3]** *and* **[A4]**.
If there exist two weak solutions $\big(X, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ *and* $\big(X', (\Omega', \mathbb{F}', \{\mathbb{F}'_t\}, \mathbb{P}'),\ W'\big)$ *to equation* (9.7) *with the same initial data* $x \in E$ *and with paths in* $C([0, T]; E)$ $\mathbb{P}$*-a.s., such that*

$$\mathbb{P}\{\textstyle\int_0^T \|\Gamma(X(s))\|_H^2 ds < \infty\} = \mathbb{P}'\{\int_0^T \|\Gamma(X'(s))\|_H^2 ds < \infty\} = 1, \tag{9.8}$$

then the laws of X *and* X' *are the same.*

For the equivalence of the laws we have

Theorem 9.2 *Assume* **[A3]** *and* **[A4]**.
Given $x \in E$, *if there exists a weak solution* $\big(X, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ *to equation* (9.7) *with paths in* $C([0,T]; E)$ $\mathbb{P}$*-a.s., and satisfying*

$$\mathbb{P}\{\int_0^T \|\Gamma(X(s))\|_H^2 ds < \infty\} = 1,$$

then there is uniqueness in law for equation (9.7) *and* $\mu^{A+F} \prec \mu^A$. *Further, if the strong solution* $\big(Z, (\Omega, \mathbb{F}, \{\mathbb{F}_t\}, \mathbb{P}),\ W\big)$ *to equation* (9.6) *satisfies*

$$\mathbb{P}\{\int_0^T \|\Gamma(Z(s))\|_H^2 ds < \infty\} = 1,$$

then $\mu^{A+F} \sim \mu^A$; *in particular*

$$\frac{d\mu^{A+F}}{d\mu^A}(Z) = \mathbb{E}\Big[e^{+\int_0^T {}_H\langle \Gamma(Z(s)), dW(s)\rangle_H - \frac{1}{2}\int_0^T \|\Gamma(Z(s)\|_H^2 ds} |\mathbb{F}_T(Z)\Big],$$

$$\frac{d\mu^{A}}{d\mu^{A+F}}(X) = \mathbb{E}\Big[e^{-\int_0^T {}_H\langle \Gamma(X(s)), dW(s)\rangle_H - \frac{1}{2}\int_0^T \|\Gamma(X(s)\|_H^2 ds} |\mathbb{F}_T(X)\Big],$$

$\mathbb{P}$*-a.s.*

Remark 9.3 *Conditions* (9.5) *become*

$$\mathbb{P}\{\int_0^T \|(\sqrt{Q})^{-1} AX(s)\|_H^2\ ds < \infty\} = 1$$

$$\mathbb{P}\{\int_0^T \|(\sqrt{Q})^{-1} AZ(s)\|_H^2\ ds < \infty\} = 1.$$

We point out that they are not satisfied in the example of Section 4 in [5].

We conclude analyzing a consequence of the equivalence of the laws. If $\mu^{A+F} \sim \mu^A$, then for any fixed $t \in [0,T]$ the law of $X(t)$ is equivalent to the law of $Z(t)$. If we know properties of the law of $Z(t)$, then they hold a.s. also for $X(t)$. Usually, properties of the solutions of the linear equation (9.6) are easier to obtain than for the non linear equation (9.7). The Girsanov transform allows to link these results. An important application is in the study of the asymptotic behaviour, as $t \to \infty$, of equation (9.7) in an infinite dimensional space (see, e.g., [4] for the general theory and examples, and [5] for examples) when our results hold on any finite time interval $[0,T]$, that is for any $T > 0$.

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